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Representations of Linear Groups

An Introduction Based on Examples from Physics and Number Theory

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Preface

There are already many good books on representation theory for all kinds of groups. Two of the best (in this author's opinion) are the one by A.W. Knapp: "Representation Theory for Semisimple Groups. An Overview based on Examples" [Kn1] and by G.W. Mackey: "Induced Representations in Physics, Probability and Number Theory" [Ma1]. The title of this text is a mixture of both these titles, and our text is meant as a very elementary introduction to these and, moreover, to the whole topic of group representations, even infinite-dimensional ones. As is evident from the work of Bargmann [Ba], Weyl [Wey] and Wigner [Wi], group representations are fundamental for the theory of atomic spectra and elementary physics. But representation theory has proven to be an inavoidable ingredient in other fields as well, particularly in number theory, as in the theory of theta functions, automorphic forms, Galois representations and, finally, the Langlands program. Hence, we present an approach as elementary as possible, having in particular these applications in mind.

This book is written as a summary of several courses given in Hamburg for students of Mathematics and Physics from the fifth semester on. Thus, some knowledge of linear and multilinear algebra, calculus and analysis in several variables is taken for granted. Assuming these prerequisites, several groups of particular interest for the applications in physics and number theory are presented and discussed, including the symmetric group $\mathfrak{S}_n$ as the leading example for a finite group, the groups SO(2), SO(3), SU(2), and SU(3) as examples of compact groups, the Heisenberg groups and $\mathrm{SL}(2,\mathbf{R})$, $\mathrm{SL}(2,\mathbf{C})$, resp. the Lorentz group $\mathrm{SO}(3,1)$ as examples for noncompact groups, and the Euclidean groups $\mathrm{E}(n) = \mathrm{SO}(n) \ltimes \mathbf{R}^n$ and the Poincaré group $\mathcal{P} = \mathrm{SO}(3,1)^+ \ltimes \mathbf{R}^4$ as examples for semidirect products.

This text would not have been possible without the assistance of my students and colleagues; it is a pleasure for me to thank them all. In particular, D. Bahns, S. Böcherer, O. v. Grudzinski, M. Hohmann, H. Knorr, J. Michaliček, H. Müller, B. Richter, R. Schmidt, and Chr. Schweigert helped in many ways, from giving valuable hints to indicating several mistakes. Part of the material was treated in a joint seminar with Peter Slodowy. I hope that a little bit of his way of thinking is still felt in this text and that it is apt to participate in keeping alive his memory. Finally, I am grateful to U. Schmickler-Hirzebruch and S. Jahnel from the Vieweg Verlag for encouragement and good advice.

Contents

Introduction

In this book, the groups enumerated in the Preface are introduced and treated as matrix groups to avoid as much as possible the machinery of manifolds, Lie groups and bundles (though some of it soon creeps in through the backdoor as the theory is further developed). Parallel to information about the structure of our groups we shall introduce and develop elements of the representation theory necessary to classify the unitary representations and to construct concrete models for these representations. As the main tool for the classification we use the *infinitesimal method* linearizing the representations of a group by studying those of the Lie algebra of the group. And as the main tools for the construction of models for the representations we use

– tensor products of the *natural* representation,

– representations given by smooth functions (in particular polynomials) living on a space provided with an action of the group,

and

– the machinery of *induced representations.*

Moreover, because of the growing importance in physics and the success in deriving branching relations, the procedure of *geometric quantization* and the *orbit method*, developed and propagated by Kirillov, Kostant, Duflo and many others shall be explained via its application to some of the examples above.

Besides the sources already mentioned, the author was largely influenced by the now classical book of Kirillov: "Elements of the Theory of Representations" [Ki] and the more recent "Introduction to the Orbit Method" [Ki1]. Other sources were the books by Barut and Raczka: "Theory of Group Representations and Applications" [BR], S. Lang: "SL$(2, \mathbf{R})$" [La], and, certainly, Serre: "Linear Representations of Finite Groups" [Se]. There is also the book by Hein: "Einführung in die Struktur- und Darstellungstheorie der klassischen Gruppen" [Hei], which follows the same principle as our text, namely to do as much as possible for matrix groups, but does not go into the infinite-dimensional representations necessary for important applications. Whoever is further interested in the history of the introduction of representation theory into the theory of automorphic forms and its development is referred to the classical book by Gelfand, Graev and Pyatetskii-Shapiro: "Representation Theory and Automorphic Forms" [GGP], Gelbart's: "Automorphic Forms on Adèle Groups" [Ge], and Bump's: "Automorphic Forms and Representations" [Bu]. More references will be given at the appropriate places in our text; as already said, we shall start using material only from linear algebra and analysis. But as we proceed more and more elements from topology, functional analysis, complex function theory, differential and symplectic geometry will be needed. We will try to introduce these as gently as possible but often will have to be very rudimentary and will have to cite the hard facts without the proofs, which the reader can find in the more refined sources.

To sum up, this text is prima facie about real and complex matrices and the nice and sometimes advanced things one can do with them by elementary means starting from a certain point of view: Representation theory associates to each matrix from a given group G another matrix or, in the infinite-dimensional case, an operator acting on a Hilbert space. One may want to ask, why study these representations by generally more complicated matrices or operators if the group is already given by possibly rather simple matrices? An answer to this question is a bit like the one for the pudding: the proof is in the eating. And we hope our text will give an answer. To the more impatient reader who wants an answer right away in order to decide whether to read on or not, we offer the following rough explanation. Certain groups G appear in nature as symmetry groups leaving invariant a physical or dynamical system. For example, the orthogonal group $O(3)$ is the symmetry group for the description of the motion of a particle in a central symmetric force field, and the Poincaré group $\mathcal{P}$ is the symmetry group for the motion of a free particle in Minkowski space. Then the irreducible unitary representations of G classify indivisible intrinsic descriptions of the system and, boldly spoken, can be viewed as "elementary particles" for the given situation. Following Wigner and his contemporaries, the parameters classifying the representations are interpreted as quantum numbers of these elementary particles...

The importance of representations for number theory is even more difficult to put into a nutshell. In the Galois theory of algebraic number fields (of finite degree) Galois groups appear as symmetry groups G. Important invariants of the fields are introduced via certain zeta- or L-functions, which are constructed using finite-dimensional representations of these Galois groups. Another aspect comes from considering smooth (holomorphic resp. meromorphic) functions in several variables which are periodic or have a more general covariant transformation property under the action of a given discrete subgroup of a continuous group G, like for instance $G = \mathrm{SL}(2, \mathbf{R})$ or G a Heisenberg or a symplectic group. Then these functions with preassigned types, e.g., theta functions or modular forms, generate representation spaces for (infinite-dimensional) representations of the respective group G.

Finally, we will give an overview over the contents of our text: In a *prologue* we will fix some notation concerning the groups and their actions that we later use as our first examples, namely, the general and special linear groups over the real and complex numbers and the orthogonal and unitary groups. Moreover, we present the symmetric group $\mathfrak{S}_n$ of permutations of n elements and some facts about its structure. We stay on the level of very elementary algebra and stick to the principle to introduce more general notions and details from group theory only when needed in our development of the representation theory. We follow this principle in the first chapter where we introduce the concept of linear representations using only tools from linear algebra. We define and discuss the fundamental notions of equivalence, irreducibility, unitarity, direct sums, tensor product, characters, and give some first examples.

The theory developed thus far is applied in the second chapter to the representations of finite groups, closely following Serre's exposition [Se]. We find out that all irreducible representations may be unitarized and are contained in the regular representation.

In the next step we move on to compact groups. To do this we have to leave the purely algebraic ground and take in topological considerations. Hence, in the third chapter, we define the notion of a topological and of a (real or complex) linear group, the central notion for our text. Following this, we refine the definition of a group representation by adding the usual continuity condition. Then we adequately modify the general concepts of the first chapter. We try to take over as much as possible from finite to compact groups. This requires the introduction of invariant measures on spaces with a (from now on) continuous group action, and a concept of integration with respect to these measures. In the forth chapter we concentrate on compact groups and prove that the irreducible representations are again unitarizable, finite-dimensional, fixed by their characters and contained in the regular representation. But their number is in general not finite, in contrast to the situation for finite groups. We state, but do not prove, the Peter-Weyl Theorem. But to get a (we hope) convincing picture, we illustrate it by reproducing Wigner's discussion of the representaions of SU(2) and SO(3). We use and prove that SU(2) is a double cover of SO(3). Angular momentum, magnetic and spin quantum numbers make an appearance, but for further application to the theory of atomic spectra we refer to [Wi] and the physics literature.

In a very short fifth chapter, we assemble some material about the representations of locally compact abelian groups. We easily get the result that every unitary irreducible representation is one-dimensional. But as can be seen from the example $G = \mathbf{R}$, their number need not be denumerable. More functional analysis than we can offer at this stage is needed to decompose a given reducible representation into a *direct integral* of irreducibles, a notion we not consider here.

Before starting the discussion of representations of other noncompact groups, we present in chapter 6 an important tool for the classification of representations, the *infinitesimal method*. Here, at first, we have to explain what a Lie algebra is and how to associate one to a given linear group. Our main ingredient is the matrix exponential function and its properties. We also reflect briefly on the notion of representations of Lie algebras. Here again we are on purely algebraic, at least in examples, easily accessible ground. We start giving examples by defining the derived representation $d\pi$ of a given group representation π. We do this for the Schrödinger representation of the Heisenberg group and the standard representation π_1 of SU(2). Then we concentrate on the classification of all unitary irreducible representations of $\mathrm{SL}(2, \mathbf{R})$ via a description of all (integrable) representations of its Lie algebra. Having done this, we consider again the examples $\mathfrak{su}(2)$ and $\mathrm{heis}(\mathbf{R})$ (relating them to the theory of the *harmonic oscillator*) and give some hints concerning the general structure theory of *semisimple Lie algebras*. The way a general classification theory works is explained to some extent by considering Lie SU(3); we will see how *quarks* show up.

Chapters 7 and 8 are the core of our book. In the seventh chapter we introduce the concept of *induced representations*, which allows for the construction of (sometimes infinite-dimensional) representations of a given group G starting from a (possibly one-dimensional) representation of a subgroup H of G. To make this work we need again a bit more Hilbert space theory and have to introduce quasi-invariant measures on spaces with group action. We illustrate this by considering the examples of the Heisenberg group and $G = \mathrm{SU}(2)$, where we rediscover the representations which we already know. Then we use the induction process to construct models for the unitary representations of

$\mathrm{SL}(2,\mathbf{R})$ and $\mathrm{SL}(2,\mathbf{C})$. In particular, we show how *holomorphic induction* arises in the discussion of the discrete series of $\mathrm{SL}(2,\mathbf{R})$ (here we touch complex function theory). We insert a brief discussion of the Lorentz group $G^L = \mathrm{SO}(3,1)^0$ and prove that $\mathrm{SL}(2,\mathbf{C})$ is a double cover of G^L. To get a framework for the discussion of the representations of the Poincaré group G^P, which is a semidirect product of the Lorentz group with $\mathbf{R}^4$, we define semidirect products and treat Mackey's theory in a rudimentary form. We outline a recipe to classify and construct irreducible representations of semidirect products if one factor is abelian. We do not prove the general validity of this procedure as Mackey's *Imprimitivity Theorem* is beyond the scope of our book, but we apply it to determine the unitary irreducible representations of the Euclidean and the Poincaré group, which are fundamental for the classification of elementary particles.

Under the heading of *Geometric Quantization*, in the eighth chapter we take an alternative approach to some material from chapter 7 by constructing representations via the *orbit method*. Here we have to recall (or introduce) more concepts from higher analysis: manifolds and bundles, vector fields, differential forms, and in particular the notion of a symplectic form. We can again use the information and knowledge we already have of our examples $G = \mathrm{SL}(2,\mathbf{R})$, $\mathrm{SU}(2)$ and the Heisenberg group to get a feeling what should be done here. We identify certain spheres and hyperboloids as coadjoint orbits of the respective groups, and we construct line bundles on these orbits and representation spaces consisting of *polarized* sections of the bundles.

Finally, in the nineth and last chapter, we give a brief outlook on some examples where representations show up in number theory. We present the notion of an automorphic representation (in a rudimentary form) and explain its relation with theta functions and automorphic forms. We have a glimpse upon Hecke's and Artin's L-functions and mention the Artin conjecture.

We hope that some of the exercises and/or omitted proofs may give a starting point for a bachelor thesis, and also that this text motivates further studies in a master program in theoretical physics, algebra or number theory.

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J. R. Jiménez

Chapter 0

Prologue: Some Groups and their Actions

This text is mainly on groups which some way or another come from physics and/or number theory and which can be described in form of a real or complex matrix group.

0.1 Several Matrix Groups

We will use the following notation:
The letter **K** indicates a field. The reader is invited to think of the field **R** of real or **C** of complex numbers. Most of the things we do at the beginning of our text are valid also for more general fields at least if they are algebraically closed and of characteristic zero, but as this is only an introduction for lack of space we will not go into this to a greater depth.
$M_{m,n}(\mathbf{K})$ denotes the **K**-vector space of $m \times n$ matrices $A = (a_{ij})$ with $a_{ij} \in \mathbf{K}$ ($i = 1, \dots, m, j = 1, \dots, n$) and $M_n(\mathbf{K})$ stands for $M_{n,n}(\mathbf{K})$.
Our groups will be (for some n) subgroups of of the *general linear group* of invertible $n \times n$-matrices

$$\mathrm{GL}(n, \mathbf{K}) := \{A \in M_n(\mathbf{K}); \det A \neq 0\}.$$

As usual, we will denote the *special linear group* by

$$\mathrm{SL}(n, \mathbf{K}) := \{A \in M_n(\mathbf{K}); \det A = 1\},$$

the *orthogonal group* by

$$\mathrm{O}(n) := \{A \in M_n(\mathbf{R}); {}^tAA = E_n\},$$

resp. for $n = p + q$

$$\mathrm{O}(p, q) := \{A \in M_n(\mathbf{R}); {}^tAD_{p,q}A = D_{p,q}\},$$

where $D_{p,q}$ is the diagonal matrix having p times 1 and q times -1 in its diagonal, and the *unitary group*

$$\mathrm{U}(n) := \{A \in M_n(\mathbf{C}); {}^tA\bar{A} = E_n\},$$

resp. for $n = p + q$

$$\mathrm{U}(p,q) := \{A \in M_n(\mathbf{R});\ {}^tAD_{p,q}\bar{A} = D_{p,q}\}.$$

Again, addition of the letter S to the symbol of the group indicates that we take only matrices with determinant 1, e.g.

$$\mathrm{SO}(n) := \{A \in \mathrm{O}(n);\ \det A = 1\}.$$

These groups together with some other families of groups, in particular the *symplectic groups* showing up later, are known as *classical groups*.

Later on, we will often use subgroups consisting of certain types of block matrices, e.g. the *group of diagonal matrices*

$$A_n := \{D(a_1,\dots,a_n);\ a_1,\dots,a_n \in \mathbf{K}^*\},$$

where $D(a_1,\dots,a_n)$ denotes the diagonal matrix with the elements $a_1,\dots,a_n$ in the diagonal, or the *group of upper triangular matrices* (or *standard Borel group*) B_n consisting of matrices with zeros below the diagonal and the *standard unipotent group* N_n, the subgroup of B_n where all diagonal elements are 1.
In view of the importance for applications, moreover, we distinguish several types of *Heisenberg groups*: Thus the group N_3 we just defined, is mostly written as

$$\mathrm{Heis}'(\mathbf{K}) := \{\ g = \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix};\ \ x,y,z \in \mathbf{K}\}.$$

In the later application to theta functions it will become clear that, though it may seem more complicated, we shall better use the following description for the Heisenberg group.

$$\mathrm{Heis}(\mathbf{K}) := \{\ g = (\lambda,\mu,\kappa) := \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix};\ \mu,\lambda,\kappa \in \mathbf{K}\ \},$$

and the "higher dimensional" groups, which for typographical reasons we here do not write as matrix groups

$$\mathrm{Heis}(\mathbf{K}^n) := \{\, g = (x,y,z);\ x,y \in \mathbf{K}^n, z \in \mathbf{K}\,\}$$

with the multiplication law given by

$$gg' = (x + x', y + y', z + z' + {}^txy' - {}^tyx').$$

We suggest to write elements of $\mathbf{K}^n$ as columns.

Exercise 0.1: Write $\mathrm{Heis}(\mathbf{K}^n)$ in matrix form. Show that $\mathrm{Heis}(\mathbf{K}^n)$ for $n = 1$ and the other two Heisenberg groups above are isomorphic.

Exercise 0.2: Verify that the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix},\ z \in \mathbf{C}^* := \mathbf{C} \setminus \{0\}$$

generate (by taking all possible finite products) a non-abelian subgroup of $\mathrm{GL}(2,\mathbf{C})$. This is called the *Weil group of* $\mathbf{R}$ and plays an important role in the epilogue at the end of our text.

0.2 Group Actions

Most groups "appear in nature" as *transformation groups* acting on some set or space. This motivates the introduction of the following concepts.

Let G be a group with neutral element e and let $\mathcal{X}$ be a set.

Definition 0.1: *G acts on $\mathcal{X}$ from the left* iff a map

$$G \times \mathcal{X} \longrightarrow \mathcal{X}\,, \quad (g,x) \longmapsto g \cdot x$$

is given, which satisfies the conditions

$$g \cdot (g' \cdot x) = (gg') \cdot x, \;\; e \cdot x = x$$

for all $g, g' \in G$ and $x \in \mathcal{X}$.

Remark 0.1: If $\operatorname{Aut}\mathcal{X}$ denotes the group of all bijections of $\mathcal{X}$ onto itself, the definition says that we have a group homomorphism $G \longrightarrow \operatorname{Aut}\mathcal{X}$ associating to every $g \in G$ the transformation $x \longmapsto g \cdot x$.

In this case the set $\mathcal{X}$ is also called a *left G-set.*

The group action is called *effective* iff no element except the neutral element e acts as the identity, i.e. the homomorphism $G \longrightarrow \operatorname{Aut}\mathcal{X}$ is faithful.

The group action is called *transitive* iff for every pair $x, x' \in \mathcal{X}$ there is a $g \in G$ with $x' = g \cdot x$.

For $x_0 \in \mathcal{X}$ we call the subset of $\mathcal{X}$

$$G \cdot x_0 := \{g \cdot x_0;\; g \in G\}$$

an *orbit of G* (through x_0) and the subgroup

$$G_{x_0} := \{g \in G;\, g \cdot x_0 = x_0\}$$

the *isotropy group* or the *stabilizing group* of x_0.

Example 0.1: $G = \mathrm{GL}(n, \mathbf{K})$ and its subgroups act on $\mathcal{X} = \mathbf{K}^n$ from the left by matrix multiplication

$$(A, x) \longmapsto Ax$$

for $x \in \mathbf{K}^n$ (viewed as a column).

Exercise 0.3: Assure yourself that $\mathrm{GL}(n, \mathbf{K})$ acts transitively on $\mathcal{X} = \mathbf{K}^n \setminus \{0\}$. Describe the orbits of $\mathrm{SO}(n)$.

Example 0.2: A group acts on itself from the left in three ways:
a) by left translation $(g, g_0) \longmapsto g \cdot g_0 = gg_0 =: \lambda_g g_0$,
b) by (the inverse of) right translation $(g, g_0) \longmapsto g \cdot g_0 = g_0 g^{-1} =: \rho_{g^{-1}} g_0$,
c) by conjugation $(g, g_0) \longmapsto g \cdot g_0 = gg_0 g^{-1} =: \kappa_g g_0$.
Left- and right translations are obviously transitive actions. For a given group, the determination of its *conjugacy classes*, i.e. the classification of the orbits under conjugation is a highly interesting question, as we shall see later.

Exercise 0.4: Determine a family of matrices containing exactly one representative for each conjugacy class of $G_1 = \mathrm{GL}(2, \mathbf{C})$ and $G_2 = \mathrm{GL}(2, \mathbf{R})$.

Definition 0.2: A set is called a *homogeneous space* iff there is a group G acting transitively on $\mathcal{X}$.

Remark 0.2: In this case one has $\mathcal{X} = G \cdot x$ for each $x \in \mathcal{X}$ and any two stabilizing groups G_x and $G_{x'}$ are conjugate. (Prove this as **Exercise 0.5**.)

Definition 0.3: For any G-set we shall denote by $\mathcal{X}/G$ or $\mathcal{X}_G$ the set of G-orbits in $\mathcal{X}$ and by $\mathcal{X}^G$ the set of *fixed points for* G, i.e. those points $x \in \mathcal{X}$ for which one has $g \cdot x = x$ for all $g \in G$.

If the set $\mathcal{X}$ has some structure, for instance, if $\mathcal{X}$ is a vector space, we will (tacidly) additionally require that a group action preserves this structure, i.e. in this case that $x \longmapsto g \cdot x$ is a linear map for each $g \in G$ (or later on, is continuous if $\mathcal{X}$ is a topological space). It is a fundamental question whether the orbit space $\mathcal{X}/G$ inherits the same properties as the space $\mathcal{X}$ may have. For instance, if $\mathcal{X}$ is a manifold, is this true also for $\mathcal{X}/G$? We will come back to this question later several times. Here let us only look at the following situation: Let $\mathcal{X}$ be a homogeneous G-space, $x_0 \in \mathcal{X}$, and $H = G_{x_0}$ the isotropy group. Then we denote by G/H the set of *left cosets* $gH := \{gh; h \in H\}$. These are also to be seen as H-orbits $H \cdot g$ where H acts on G by right translation. G/H is a G-set, G acting by $(g, g_0 H) \longmapsto gg_0 H$. Then we shall identify G/H and $\mathcal{X}$ via the map

$$G/H \longrightarrow \mathcal{X}, \quad gH \longmapsto g \cdot x_0.$$

This map is an example for the following general notion.
Definition 0.4: Let $\mathcal{X}$ and $\mathcal{X}'$ be G-sets and $f : \mathcal{X} \longrightarrow \mathcal{X}'$ be a map. The map f is called G-*equivariant* or a G-*morphism* iff one has for every $g \in G$ and $x \in \mathcal{X}$

$$g \cdot f(x) = f(g \cdot x).$$

Now we come back to the question raised above.
Exercise 0.6: Prove that G/H has the structure of a group iff H is not only a subgroup but a *normal* subgroup, i.e. one has $ghg^{-1} \in H$ for all $g \in G, h \in H$.

We mention here another useful fact: If H is a subgroup of G, one defines the *normalizer* of H in G as

$$N_G(H) := \{g \in G;\ gHg^{-1} = H\}.$$

It is clear that $N_G(H)$ is the maximal subgroup in G that has H as a normal subgroup. Then the group $\mathrm{Aut}\,(G/H)$ of G-equivariant bijections of G/H is isomorphic to $N_G(H)/H$. (Prove this as **Exercise 0.7**.)