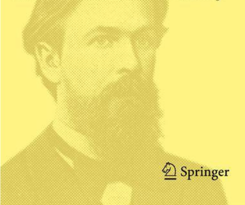
Martin Aigner

# Markov's Theorem and 100 Years of the Uniqueness Conjecture

## A Mathematical Journey from Irrational Numbers to Perfect Matchings



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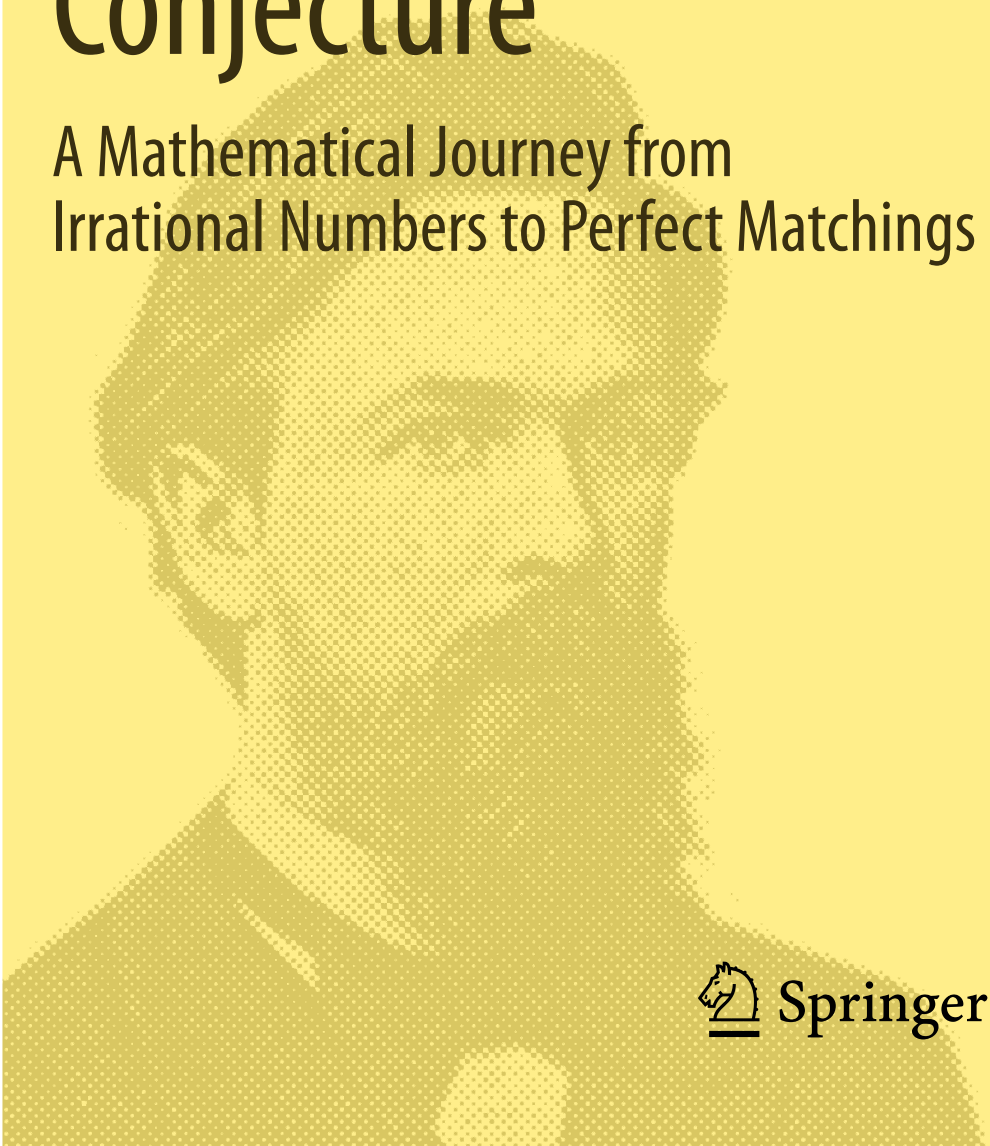

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# Preface

When you ask different people what they like most about mathematics, you are likely to get different answers: For some, it is the clarity and beauty of pure reasoning; for others, grand theories or challenging problems; and for others still the elegance of proofs. For me – ever since my student days – mathematics has its finest moment when a concept or theorem turns up in seemingly unrelated fields, when ideas from different parts come together to produce a new and deeper understanding, when you marvel at the unity of mathematics.

This book seeks to describe one such glorious moment. It tells the story of a celebrated theorem and an intriguing conjecture: Markov's theorem from 1879 and the uniqueness conjecture first mentioned by Frobenius precisely 100 years ago in 1913. I first learned about Markov's theorem some thirty years ago and was immediately captivated by this stunning result, which combines approximation of irrationals and Diophantine equations in a totally unexpected way. It must have struck the early researchers in a similar manner. Georg Frobenius writes in his treatise from 1913: "*Trotz der außerordentlich merkwürdigen und wichtigen Resultate scheinen diese schwierigen Untersuchungen wenig bekannt zu sein.*" [In spite of the extraordinarily strange and important results these difficult investigations seem to be little known.] Frobenius did his best to remedy the situation, and it was in this paper that he mentioned what has become the uniqueness conjecture.

In the 100 years since Markov's theorem, Markov numbers, which play a decisive role in the theorem, have turned up in an astounding variety of different settings, from number theory to combinatorics, from classical groups and geometry to the world of graphs and words. The theorem has become a classic in number theory proper, but at least judging from my own experience, neither the theorem nor the conjecture, let alone the many beautiful interconnections, is as well known in the general mathematical community as it deserves. It is the aim of this book to present an up-to-date and fairly complete account of this wonderful topic.

True to the philosophy of bringing different fields together, this book is arranged in a somewhat nonstandard way. It consists of five parts; each containing two chapters: Numbers, Trees, Groups, Words, Finale. The first part sets the stage, introducing Markov's theorem and the uniqueness conjecture.

The proof of the theorem and an account of the present state of the conjecture are, however, postponed until the last part. The three parts in-between describe in detail the various fields in which the theorem and conjecture turn up, sometimes in quite mysterious ways.

Of course, one could proceed more directly to a proof of the theorem, but the present approach permits one to look at both the theorem and conjecture from many different viewpoints, gradually getting to the heart of the Markov theme until everything falls into place. And there is an additional bonus: The three middle parts present introductory courses on some beautiful mathematical topics that do not usually belong to the standard curriculum, including Farey fractions, the modular and free groups, the hyperbolic plane, and algebraic words.

Here is a short overview of what this book is all about. Part I (Numbers) starts with the fundamental question of best approximation of irrationals by rationals, leading to the Lagrange number and the Lagrange spectrum. On the way, we encounter some of the all-time classics of number theory such as continued fractions and the theorems of Dirichlet, Liouville, and Roth about the limits of approximation. In Chapter 2, the Markov equation and Markov numbers are introduced, setting the stage for the statement of Markov's theorem and the uniqueness conjecture.

Part II (Trees) begins the study of Markov numbers. It turns out that all Markov numbers, or more precisely Markov triples, can be obtained by a simple recurrence. The natural dataset to encode this recurrence is to regard the Markov triples as vertices of an infinite binary tree, the Markov tree. The study of this tree is fun and allows easy proofs of some basic results, e.g., that all odd-indexed Fibonacci numbers are Markov numbers. Next, we take up another evergreen of number theory, Farey fractions, and observe that they can also be arranged in a binary tree. This Farey tree is then used to index the Markov numbers, a brilliant idea of Frobenius that represented one of the early breakthroughs. Another step forward is made in Chapter 4, where it is shown that Markov numbers appear rather unexpectedly in a certain class of integral $2 \times 2$ matrices, named Cohn matrices after one of their first proponents. In this way, a third tree is obtained that permits elegant proofs of further properties of Markov numbers.

Part III (Groups) rightly occupies the central place in the exposition. It describes, via Cohn matrices, the surprising and beautiful connections of the Markov theme to the classical groups $\mathrm{SL}(2, \mathbb{Z})$ and $\mathrm{GL}(2, \mathbb{Z})$, to the Poincaré model of the hyperbolic plane, to Riemann surfaces, and to the free group on two generators. This is perhaps the mathematically most interesting part, and to enable the reader to appreciate it fully, there will be a concise introduction to the modular group $\mathrm{SL}(2, \mathbb{Z})$ in Chapter 5 and to free groups in Chapter 6. At the end, we will see that Cohn matrices may be equivalently regarded as finite strings over an alphabet of size 2, say $\{A, B\}$, handing the Markov theme from algebra over to combinatorics.

Part IV (Words) explores this combinatorial setting, containing both classical and fairly new results about Markov numbers. To every Farey fraction is associated a lattice path in the plane grid with a natural encoding over $\{A, B\}$, and these words turn out to be exactly the Cohn words. This fundamental result was actually known (to Christoffel and others) before Markov and was a source of inspiration for his own work. Rather recent is the discovery that the Markov numbers indeed count something, namely the number of matchings of certain graphs embedded in the plane grid. This is not only remarkable but yields a new and promising algorithmic version of the uniqueness conjecture. Chapter 8 takes the final step towards the proof of Markov's theorem. It relates continued fractions and the Lagrange spectrum to a special class of infinite words, Sturmian words. The chapter contains a brief introduction to their beautiful theory, including the celebrated theorem of Morse and Hedlund, and then goes on to the aspects most relevant to the Markov theme.

Part V (Finale) returns to our original task and brings all things together. Markov's theorem is proved in Chapter 9, and in the last chapter, all versions of the uniqueness conjecture that were encountered during the journey through this book are laid out once more, together with various numerical, combinatorial, and algebraic ideas that have been advanced in quest of a proof of the conjecture.

The requirements for reading and enjoying everything in this book are relatively modest. Most of the material should be accessible to upper-level undergraduates who have learned some number theory and maybe basic algebra. Anything that goes beyond this level is fully explained in the text, e.g., the necessary concepts about groups and geometry in Part III, about the theory of words in Part IV, or about algebraic number theory in the last chapter.

This is not a textbook in the usual sense of laying the foundations of a particular discipline. Instead, it seeks to narrate the story of an especially beautiful mathematical discovery in one field and its many equally beautiful manifestations in others. The book is written in most parts at a leisurely pace; each chapter begins with a road map of what is ahead and ends with some historical remarks, references to the literature, and recommended further reading.

A preliminary version has been used as source material for several undergraduate seminars. In fact, it was precisely the positive response of the students to this interdisciplinary look at mathematics and their curiosity to learn “how everything hangs together with everything else” that suggested putting these notes into polished form and making them available to a larger public.

I am grateful to many colleagues, friends, and students, in particular to Dennis Clemens, to Margrit Barrett and Christoph Eyrich for the superb technical work and layout, to David Kramer for his meticulous copyediting, and to Mario Aigner and Joachim Heinze of Springer-Verlag for their interest and enjoyable cooperation.

Writing this book has been great fun, and it is my hope that its readers will equally appreciate the “Markov theme” as a great mathematical achievement and at the same time as an intellectual pleasure.

Berlin, Spring 2013 Martin Aigner

## Picture Credits

The portraits of Frobenius, Hurwitz, and Morse are frontispieces of their respective Collected Works:

> Ferdinand Georg Frobenius, Gesammelte Abhandlungen, edited by J.-P. Serre, Springer 1968,
>
> Mathematische Werke von Adolf Hurwitz, edited by Eidgenössische Technische Hochschule Zurich, Birkhäuser 1932,
>
> Marston Morse, Selected Papers, edited by R. Bott, Springer 1981.

The pictures of Dirichlet, Liouville, Markov, Roth, and Sturm are from the MacTutor History of Mathematics archive.

The portrait of Christoffel is reproduced from the ETH Zurich archives, with permission.

The photo of Bombieri was taken by John Petter Reinertsen and is reproduced with his kind permission.

The picture of Harvey Cohn was taken at a meeting of the American Mathematical Society honoring his sixty years as a member of the AMS. Despite some efforts the author was unable to contact Professor Cohn and hopes that he would give his consent.

# Contents

# I Numbers

# 1 Approximation of Irrational Numbers

## 1.1 Lagrange Spectrum

Our story begins with one of the oldest questions in number theory: How well can a real number be approximated by rational numbers? Phrased in this way, the answer is "arbitrarily well," since every real number $\alpha$ is the limit of a sequence $(\frac{p_n}{q_n})$ of rationals. But in such a convergent sequence, e.g., the decimal expansion of an irrational number $\alpha$, the denominators usually grow very fast. So let us reformulate the question as follows: Are there rational numbers $\frac{p}{q}$ close to $\alpha$, maybe infinitely many, with comparatively small denominator $q$?

A word on notation: Whenever $\frac{p}{q}$ is a rational number, it is tacitly assumed that the denominator $q$ is positive.

A first answer is provided by a classical theorem of Dirichlet.

**Theorem 1.1.** *Let $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$. There exists $\frac{p}{q} \in \mathbb{Q}$ with $q \leq N$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qN} \quad \left(\leq \frac{1}{q^2}\right).$$

***Johann Peter Gustav Lejeune Dirichlet*** *was born in 1805 in Düren, First French Empire, today in Germany. He studied in Paris and spent some time there as a tutor. With the support of Gauss and Alexander von Humboldt he was offered a position in Berlin, where he remained for almost 30 years. In 1855, he was appointed as successor of Gauss in Göttingen, where he died in 1859. His main research interest was number theory, but he also made lasting contributions to analysis and mathematical physics. In number theory he is best known for his theorem on the infinitude of primes in arithmetic progressions, the introduction of Dirichlet characters, L-functions, and the class number formula.*

M. Aigner, *Markov's Theorem and 100 Years of the Uniqueness Conjecture: A Mathematical Journey from Irrational Numbers to Perfect Matchings*, DOI 10.1007/978-3-319-00888-2_1,

*Proof.* For real $\beta$ let us write $\beta = \lfloor \beta \rfloor + \{\beta\}$, where $\lfloor \beta \rfloor$ is the integer part, and $0 \le \{\beta\} < 1$. Look at the partition

$$[0,1) = \bigcup_{i=0}^{N-1} \left[\frac{i}{N}, \frac{i+1}{N}\right)$$

of the interval $[0,1)$ into $N$ parts, and consider the $N+1$ numbers $\{\alpha\}, \{2\alpha\}, \dots, \{(N+1)\alpha\}$. By the pigeonhole principle, there are integers $k, \ell$, $1 \le k < \ell \le N+1$, such that $\{k\alpha\}$ and $\{\ell\alpha\}$ lie in the same part, whence $|\{\ell\alpha\} - \{k\alpha\}| < \frac{1}{N}$. Setting $q = \ell - k \le N$, $p = \lfloor \ell\alpha \rfloor - \lfloor k\alpha \rfloor$, we have

$$q\alpha = \ell\alpha - k\alpha = \lfloor \ell\alpha \rfloor - \lfloor k\alpha \rfloor + \{\ell\alpha\} - \{k\alpha\} = p + \{\ell\alpha\} - \{k\alpha\},$$

and thus

$$|q\alpha - p| = |\{\ell\alpha\} - \{k\alpha\}| < \frac{1}{N}.$$

Division by $q$ yields the result. □

We can immediately strengthen Dirichlet's theorem, thereby separating rationals from irrationals.

**Corollary 1.2.** 1. *If $\alpha \notin \mathbb{Q}$, then there are infinitely many $\frac{p}{q} \in \mathbb{Q}$ with $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$.*

2. *For $\alpha = \frac{r}{s} \in \mathbb{Q}$, the inequality $\left|\alpha - \frac{p}{q}\right| < \frac{C}{q^2}$ is, for every $C > 0$, satisfied for only finitely many $\frac{p}{q} \in \mathbb{Q}$.*

*Proof.* (1) According to the theorem, there exists $\frac{p_n}{q_n} \in \mathbb{Q}$ with $|\alpha - \frac{p_n}{q_n}| < \frac{1}{nq_n} \le \frac{1}{q_n^2}$ for every $n \in \mathbb{N}$. If there were only finitely many different $\frac{p_n}{q_n}$, let $\frac{p_k}{q_k}$ be such that

$$\left|\alpha - \frac{p_k}{q_k}\right| \le \left|\alpha - \frac{p_n}{q_n}\right| \quad \text{for all } n.$$

Since $\alpha \notin \mathbb{Q}$, we have $\alpha \neq \frac{p_k}{q_k}$, and hence for large enough $n$,

$$\frac{1}{n} < \left|\alpha - \frac{p_k}{q_k}\right| \le \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{nq_n} \le \frac{1}{n},$$

which cannot be.

(2) Suppose $\alpha = \frac{r}{s}$ and consider $\frac{p}{q} \neq \frac{r}{s}$ with $\left|\frac{r}{s} - \frac{p}{q}\right| < \frac{C}{q^2}$. This implies

$$\frac{1}{sq} \le \left|\frac{rq - sp}{sq}\right| = \left|\frac{r}{s} - \frac{p}{q}\right| < \frac{C}{q^2},$$

that is, $q < sC$. Hence there are only finitely many possible denominators $q$, and for a fixed $q$, only finitely many $p$. □