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The Fields Institute for Research in Mathematical Sciences

Philippe Guyenne
David Nicholls
Catherine Sulem
Editors

Hamiltonian Partial Differential Equations and Applications

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The Fields Institute for Research in Mathematical Sciences

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Hamiltonian Partial Differential Equations and Applications

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Editors
Philippe Guyenne
Department of Mathematical Sciences
University of Delaware
Newark, DE, USA

Catherine Sulem
Department of Mathematics
University of Toronto
Toronto, ON, Canada

David Nicholls
Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Chicago, IL, USA

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Preface

Partial differential equations (PDEs) are a fundamental tool in the modeling of phenomena arising in the physical sciences. PDEs with Hamiltonian structure are a distinguished subset, which not only model systems with conserved quantities (e.g., energy and momentum), but also possess an array of special techniques for their analysis and simulation. They constitute an active area of research where major innovations have been and continue to be made from both the mathematical and computational sides. Not only do these innovations benefit the field itself but also contribute to progress in a vast range of other scientific areas. Applications of Hamiltonian PDEs are numerous in fluid mechanics, plasma physics, and nonlinear optics with such notable examples as the Korteweg–de Vries equation and the nonlinear Schrödinger equation.

In the last few decades, significant progress has been achieved in the mathematical study of these evolutionary PDEs by adopting the "dynamical systems" approach, extending refined analytical techniques of Hamiltonian dynamical systems to the setting of PDEs. This point of view has led to the consideration of the global behavior of orbits for a Hamiltonian PDE in an appropriate phase space, the pursuit of the mathematical technology of normal forms, the study of stable orbits and Kolmogorov–Arnold–Moser (KAM) tori, and a number of results analogous to Nekhoroshev stability and Arnold diffusion. In particular, building on the experience gained from the qualitative study of finite-dimensional dynamical systems, the search for periodic and quasi-periodic solutions has been regarded as a first step towards better understanding the complicated flow evolution of Hamiltonian PDEs. A central tool is transformation theory including Birkhoff normal form transformations. In the broad picture, the goal is to understand some of the important structures of infinite-dimensional phase spaces in which these evolutionary equations are naturally posed, such as periodic orbits, embedded invariant tori, center manifolds, and the different effects of resonances in the non-compact versus compact cases. Techniques from transformation theory for Hamiltonian PDEs with a small parameter have also been successfully used in recent work on water waves, allowing for the systematic derivation of Hamiltonian models in various asymptotic limits.

On 10–12 January 2014, a conference on "Hamiltonian PDEs: Analysis, Computations and Applications" was held at the Fields Institute for Research in Mathematical Sciences in Toronto, bringing together a group of world-class researchers to present and discuss the latest developments in this field. Given the wide range of applications and mathematical tools, a motivating theme of this event was the interaction of specialists in dynamical systems, KAM theory, normal form theory, PDE theory and variational methods, as well as applied and numerical analysts, and experts in water waves. The program consisted of eighteen lectures by distinguished faculty, together with three shorter presentations by junior speakers including two graduate students. The participants came from Canada, Europe, and the USA.

This conference was also an opportunity to honor our friend and colleague Walter Craig, who has made significant contributions to this field, on the occasion of his 60th birthday. Walter obtained his Ph.D. degree from the Courant Institute of Mathematical Sciences (NYU) in 1981. He has held faculty positions at CalTech, Stanford University and Brown University before joining McMaster University as a Professor and Canada Research Chair of Mathematical Analysis and its Applications. He has received a number of prestigious awards including an Alfred P. Sloan Fellowship, an NSF Presidential Young Investigator Award, a Killam Research Fellowship and is a Fellow of the AMS, the AAAS, the Fields Institute and the Royal Society of Canada. He has served on the editorial boards of several journals including the Philosophical Transactions of the Royal Society, the Proceedings of the AMS, and the SIAM Journal on Mathematical Analysis. Walter is a world-renowned mathematical analyst with interests in nonlinear PDEs, Hamiltonian dynamical systems and their physical applications. He has authored more than 100 research articles.

This special volume presents a unique selection of both survey and original research papers by experts who participated in that conference. The various topics discussed in this volume are representative of the wide scope covered by Hamiltonian PDEs, and the results range from mathematical modeling to rigorous analysis and numerical simulation. These topics also reflect Walter Craig's breadth

in research interests and his influence in this field. This book will be of particular interest to graduate students as well as researchers in mathematics, physics, and engineering, who wish to learn more about the powerful and elegant analytical techniques for Hamiltonian PDEs.

The editors would like to thank the Fields Institute for Research in Mathematical Sciences and the Department of Mathematics & Statistics at McMaster University for their generous support. In particular, we are grateful to Alison Conway, Drs. Matheus Grasselli and Hans Boden for their assistance with the organization of the conference, as well as to Debbie Iscoe, Dr. Carl Riehm, and the Springer team for their assistance with the publication of this special volume. We are also thankful to the authors for contributing such excellent articles and to the referees for their invaluable help during the review process. Finally, we dedicate this book to Walter Craig who has been a constant source of inspiration, and whose enthusiasm and friendship have never waned. We would like to extend to him our warmest wishes for many more happy events to come.

Newark, DE, USA — Philippe Guyenne
Chicago, IL, USA — David Nicholls
Toronto, ON, Canada — Catherine Sulem
April 2015

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Hamiltonian Structure, Fluid Representation and Stability for the Vlasov–Dirac–Benney Equation

Claude Bardos and Nicolas Besse

Pour Walter Craig, en remerciement pour ses contributions scientifiques et son amitié.

Abstract This contribution is an element of a research program devoted to the analysis of a variant of the Vlasov–Poisson equation that we dubbed the Vlasov–Dirac–Benney equation or in short V–D–B equation. As such it contains both new results and efforts to synthesize previous observations. One of main links between the different issues is the use of the energy of the system. In some cases such energy becomes a convex functional and allows to extend to the present problem the methods used in the study of conservation laws. Such use of the energy is closely related to the Hamiltonian structure of the problem. Hence it is a pleasure to present this article to Walter Craig in recognition to the pioneering work he made for our community, among other things, on the relations between Hamiltonian systems and Partial Differential Equations.

1 Introduction

This article extends a program (cf. [1, 2]) devoted to the mathematical analysis of an avatar of the Vlasov–Poisson equation, where the "Coulomb potential" is replaced by the Dirac mass. Since it was proposed by Benney [3] and Zakharov [28] for the description of water waves, it is dubbed Vlasov–Dirac–Benney equation (or in short V–D–B equation). Therefore the V–D–B equation reads

C. Bardos (✉)
Laboratoire Jacques-Louis Lions, Université Denis Diderot, Paris, France
e-mail: claude.bardos@gmail.com

N. Besse
Département Physique de la Matière et des Matériaux, Institut Jean Lamour UMR CNRS 7198, Université de Lorraine, BP 70239, 54506 Vandoeuvre-lès-Nancy Cedex, France
e-mail: nicolas.besse@univ-lorraine.fr

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$$\partial_t f + v \cdot \nabla_x f - \nabla_x \rho_f \cdot \nabla_v f = 0\,, \quad \text{with} \quad \rho_f(t,x) = \int_{\mathbb{R}^d} f(t,x,v)dv. \tag{1}$$

And the classical conservation of mass and energy turn out to be given by the formula,

$$\partial_t \int_{\mathbb{R}^d_v} f(t,x,v)dv + \nabla_x \cdot \int_{\mathbb{R}^d_v} v f(t,x,v)dv = 0,$$

$$\begin{aligned}\partial_t \int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} \frac{|v|^2}{2} f(t,x,v)dxdv &= -\int_{\mathbb{R}^d_x} \nabla_x \rho_f(t,x) \cdot \int_{\mathbb{R}^d_v} \frac{|v|^2}{2} \nabla_v f(t,x,v)dxdv \\ &= \int_{\mathbb{R}^d_x} \rho_f(t,x) \nabla_x \cdot \int_{\mathbb{R}^d_v} v f(t,x,v)dxdv \\ &= -\int_{\mathbb{R}^d_x} \rho_f(t,x) \partial_t \rho_f(t,x)dx,\end{aligned}$$

or eventually

$$\frac{d\mathscr{E}}{dt} = \frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^d_x} dx \left(\int_{\mathbb{R}^d_v} dv\, \frac{|v|^2}{2} f(t,x,v) + (\rho_f(t,x))^2\right)\right) = 0.$$

1.1 Some Physical Motivations for the Introduction of the Dirac Potential

One of the many physical motivations for the introduction of this equation is the description of a plasma constituted of ions in a background of "adiabatic" electrons which instantaneously reach a thermodynamical equilibrium (i.e. electrons follow a Maxwell–Boltzmann distribution). Therefore the charge density of electrons is given in term of the electrical potential Φ_ϵ by the formula

$$\rho_- = \rho_0 e^{-\frac{e\Phi_\epsilon}{k_B T_e}},$$

with k_B the Boltzmann constant, e the electron charge and T_e the equilibrium temperature of electrons. Finally the parameter ϵ represents the Debye length. Hence the "Coulomb law" couples the electrical potential Φ_ϵ to the charge density such that

$$-\epsilon^2 \Delta \Phi_\epsilon = \rho_\epsilon - \rho_0 e^{-\frac{e\Phi_\epsilon}{k_B T_e}}\,, \quad \text{with} \quad \rho_\epsilon = \int_{\mathbb{R}^d_v} f_\epsilon(t,x,v)dv.$$

Now since the electrical potential energy $e\Phi_\epsilon$ is supposed to be small in comparison to the kinetic energy $k_B T_e$, i.e $|e\Phi_\epsilon/(k_B T_e)| \ll 1$, after linearization on the exponential function, at first order we get

$$-\epsilon^2 \Delta \Phi_\epsilon = \rho_\epsilon - \rho_0 + \frac{e\rho_0}{k_B T_e}\Phi_\epsilon.$$

Setting ϵ to zero (quasineutrality assumption) and since ρ_0 and T_e are supposed to be constant, we obtain for the electric field E_ϵ the expression

$$E_\epsilon = -\nabla_x \Phi_\epsilon = \frac{k_B T_e}{e\rho_0}\nabla_x \int_{\mathbb{R}^d_v} f_\epsilon(t,x,v)dv,$$

which appears in the Vlasov equation (1).

1.2 Some Mathematical Motivations for this Analysis

Since in the Eq. (1) the electric field E is given in term of the electrons density by an operator of order 1, while in the classical Vlasov–Poisson case it is given by an operator of degree -1, the solution is much more dependent on the initial data. Therefore, while for the classical Vlasov–Poisson equation the issue is the large time asymptotic behavior, here what is at stake is the well-posedness of the problem in term of the initial data. On the other hand since the electrical potential is given by a purely local operator there exists a strong connection between the dynamics of hyperbolic systems of conservation laws and the V–D–B equation. This connection appears even more clearly when one uses for the Vlasov equation a kinetic representation of the form (cf. Sect. 3.2)

$$f(t,x,v) = \int_M \rho(t,x,\sigma)\delta(v - u(t,x,\sigma))d\sigma, \tag{2}$$

which leads to non local "operator type" conservation laws.

For such conservation laws the invariants play an essential role and as expected, they coincide (cf. Theorem 6) with the Lax–Godunov conserved quantities. When such invariants turn out to be convex (with respect to the parameters of the dynamics) they play the role of convex entropies and ensure the local-in-time stability and well-posedness of the Cauchy problem.

As this is the case for the most general Vlasov equations (as explained for instance in [22]) the present V–D–B equation can be viewed as a Hamiltonian system related to the minimization of an energy. Moreover the same point of view can be used to formalize the relations between classical and quantum mechanics via semi-classical (WKB) limits and Wigner measure (cf. Sect. 6). Such convergence will be always true at the formal level, or with analytical initial data. However, as expected, proofs in the Distributions (or Sobolev) setting will be available only when the limit enjoys the same stability i.e. mostly in the case where a convex entropy is present. Even if the analyticity hypothesis is not "physical", conclusions that follow are important, and especially in the case of the one-dimensional space

variable. Then the cubic nonlinear Schrödinger equation and its generalization as infinite systems of coupled nonlinear Schrödinger equations (cf. [28]) are integrable systems with a rich algebraic structure including in particular the construction of infinite family of conservation laws. In the semi-classical limits these structures (at least for analytic initial data) do persist and make the one-dimensional-space-variable V–D–B equation a quasi-integrable system in the sense of [28].

The paper is then organized as follows. First the emphasis is put on the one-dimensional space variable which as quoted above contains more mathematical structure and provides also more explicit examples. To underline the dimension one in the corresponding equations, the symbol ∇_x and ∇_v are replaced by the symbol ∂_x and ∂_v. In Sect. 2, the analysis of the linearized problems turns out to be (and this should not be a surprise) in full agreement with the properties of the fully nonlinear systems. Moreover this produces also a natural tool for the study of nonlinear perturbations which is the object of the next section.

In the Sect. 3, the Hamiltonian structure and the fluid representation of the kinetic V–D–B equation are described. In this setting, under strong analyticity hypothesis a local-in-time stability result can be proven and this is the object of the Theorem 5. To obtain stability results with finite order regularity, the entropies have to be introduced and compared with the classical invariants of the Hamiltonian system. This is the object of the Sect. 4 and Theorem 6. The next Sect. 5 is devoted to several examples of application.

For the discussion of the semi-classical limits in the Sect. 6, we follow similar route. First formal computations are given. Then there are validated with analyticity hypothesis (cf. Theorem 9). Such results are compared with a theorem of Grenier which is valid in any space variable, with Sobolev type regularity hypothesis, but which concerns only the Wigner limit of "pure states" i.e. mono-kinetic solutions of the V–D–B equation.

As a conclusion we return to the relation between Wigner limit and inverse scattering.

2 Properties of the Linearized Problem and Consequences

Long time ago, it has been observed that x-independent solutions

$$v \mapsto G(v) \geq 0 \quad \text{with} \quad \int_{\mathbb{R}} G(v)dv = 1,$$

are stationary solutions of the Vlasov–Poisson equation. Same simple observation is also valid for the V–D–B equation. Writing

$$f(t,x,v) = G(v) + \tilde{f}(t,x,v),$$